Correlation Studying the Maple Package of the Bessel Equation with a Random Inhomogeneous Part

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Abstract—In this paper, a correlation study of the inhomogeneous Bessel equation is carried out. These studies are implemented in the mathematical package Maple; the surfaces of the correlation function and the variance tables of partial solutionscorresponding to a given correlation function of an inhomogeneous part - either a random function or derivatives of thisrandom function are constructed. The theory of random functions and the theory of Bessel functions are mainly used. If we takeinto account that the formulas obtained for the correlation functions of solving such inhomogeneous equations are associated with great computational difficulties, then the results obtained in the Maple environment acquire specialpractical interest.

Keywords—random function, correlation function, Bessel equation, Maple.

An inhomogeneous Bessel equation with a random functionin the right part is investigated:

$$\frac{d^2}{dt^2}Y(t)+\frac{1}{t}\frac{d}{dt}Y(t)+\left(1-\frac{\nu^2}{t^2}\right)Y(t)=\frac{d^k}{dt^k}X(t),\quad k=0,1,2,\dots$$

where X(t) is a stationary random function with zero mathematical expectation and with a known correlation function; here k = 0 corresponds to the case, for example, when a signal about the instantaneous values of the coordinates of a rapidly moving object is received at the input of the system, k = 1 is the case of a speed signal, k = 2 is the case of a signal about the accelerationof motion.

To determine the correlation function of a particular solution of a linear differential equation satisfying the zero-initial condition, it is first necessary to find the corresponding weight function.

The weight function of a dynamical system described by this differential equation is determined using a fundamental system of solutions of the corresponding homogeneous equation $\{J_\nu(t),N_\nu(t)\}$ ($J_\nu(t)$ – Bessel function of the first kind, $N_\nu(t)$ – Bessel function of the second kind) ([1]):

$$p(\nu,t,t1)=\frac{[J_\nu(t1)N_\nu(t)-J_\nu(t)N_\nu(t1)]}{W[J_\nu,N_\nu](t1)}.$$

As you know, the Wroskian determinant ([2])

$$W[J_\nu,N_\nu](t1)=J_\nu(t1)N'_\nu(t)-J'_\nu(t)N_\nu(t1)=\frac{2}{\pi\cdot t1}$$

and, therefore

$$p(\nu,t,t1)=\frac{\pi}{2}t1[J_\nu(t1)N_\nu(t)-J_\nu(t)N_\nu(t1)].$$

As a result, we obtain the following formulas for a particular solution of the equation under study and the correlation function of this solution:

$$Y_k(t)=\int_0^t p(\nu,t,t1)\frac{d^k}{dt1^k}X(t1)dt1,\qquad k=0,1,2$$

$$K_{y_k}(\nu,t1,t2)=\int_0^{t1}\int_0^{t2}p(\nu,t1,s1),p(\nu,t2,s2)K_{x^{(k)}}(s1,s2)ds1ds2,\quad k=0,1,2.$$

Calculations using the latter formula are fraught with great difficulties, therefore,for practical usefulness, we carry out all further calculations in the environment ofthe mathematical package Maple ([3], [4]).

Determination of the weight function p (ν, t, t1)

> restart;

>ps:=unapply((BesselJ(nu,t1)*BesselY(nu,t)-BesselJ(nu,t)*BesselY(nu,t1)),nu,t,t1);

$$ps:=(\nu,t,t1)\mapsto BesselJ(\nu,t1)\cdot BesselY(\nu,t)-BesselJ(\nu,t)\cdot BesselY(\nu,t1)$$

>p:=unapply(simplify(ps(nu,t,t1)/(2/(Pi*t1))),nu,t,t1);

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$$p := (\nu, t, t1) \mapsto \frac{(BesselJ(\nu, t1) \cdot BesselY(\nu, t)}{2} - \frac{BesselJ(\nu, t) \cdot BesselY(\nu, t1)) \cdot \pi \cdot t1}{2}$$

Input of the correlation function (c. f.) of the input signal X(t) ([5], [6])

>Kx:=unapply(exp(-(s1-s2)^2),s1,s2);

$$Kx := (s1, s2) \mapsto e^{-|s1-s2|^2}$$

>Kxn:=unapply(exp(-abs(s1-s2))*(1+abs(s1-s2)),s1,s2);

$$Kxn := (s1, s2) \mapsto e^{-|s1-s2|} \cdot (1 + |s1 - s2|)$$

>Kxc:=unapply(exp(-(s1-s2)^2)*cos(s1-s2),s1,s2);

$$Kxc := (s1, s2) \mapsto e^{-|s1-s2|^2} \cdot \cos(s1 - s2)$$

Determination of the correlation function of a partial solution,corresponding to X(t) with correlation functionKxc and Kxn at ν = 0

**>Ky0s:=unapply((Pi^2/4)*int(int(s1*s2*p(0,s1,t1)*p(0,s2,t2)*
Kxc(s1,s2),s1=0..t1),s2=0..t2),t1,t2);**

$$Ky0s := (t1, t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} \frac{1}{4}(s1 \cdot s2 \cdot (BesselJ(0, t1) \cdot BesselY(0, s1) - BesselJ(0, s1) \cdot BesselY(0, t1)) \cdot \pi^2 \cdot t1 \cdot (BesselJ(0, t2) \cdot BesselY(0, s2) - BesselJ(0, s2) \cdot BesselY(0, t2)) \cdot t2 \cdot e^{-(s1-s2)^2} \cdot \cos(-s1 + s2)) ds1 ds2\right)\right)$$

**>
Ky0n:=unapply((Pi^2/4)*int(int(s1*s2*p(0,s1,t1)*p(0,s2,t2)*
Kxn(s1,s2),s1=0..t1),s2=0..t2),t1,t2);**

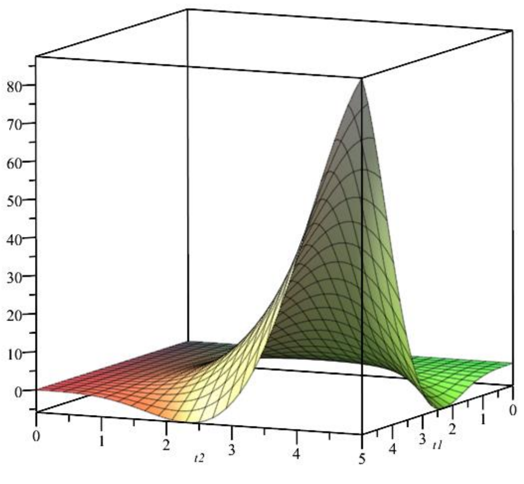

Figure 1

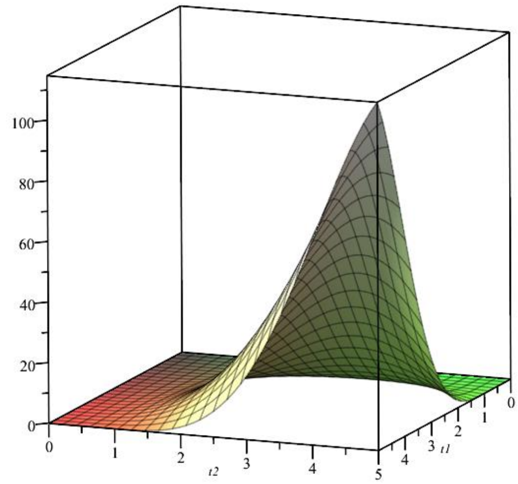

Figure 2

$$Ky0s := (t1, t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} \frac{1}{4}(s1 \cdot s2 \cdot (BesselJ(0, t1) \cdot BesselY(0, s1) - BesselJ(0, s1) \cdot BesselY(0, t1)) \cdot \pi^2 \cdot t1 \cdot (BesselJ(0, t2) \cdot BesselY(0, s2) - BesselJ(0, s2) \cdot BesselY(0, t2)) \cdot t2 \cdot e^{-|-s1+s2|} \cdot (1 + |-s1 + s2|)) ds1 ds2\right)\right)$$

> plot3d(Ky0s(t1,t2),t1=0..5,t2=0..5);

> plot3d(Ky0n(t1,t2),t1=0..5,t2=0..5);

>evalf([Ky0s(5,5),Ky0n(5,5)]);

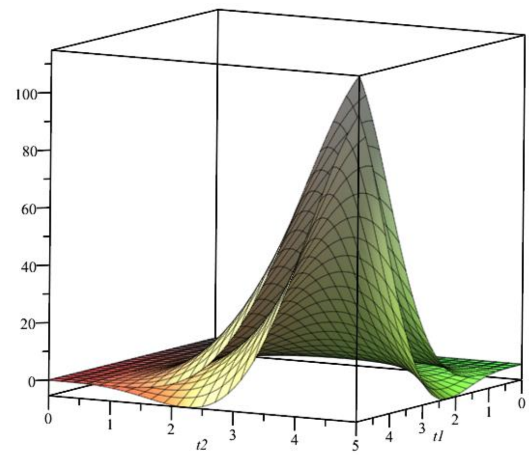

Figure 3

[87.40943122, 114.7107820]

Determination of the correlation function of apartial solution,corresponding to X(t) with correlation functionKx at ν = 0

> Ky0:=unapply((Pi^2/4)*int(int(s1*s2*p(0,s1,t1)*p(0,s2,t2)* Kx(s1,s2),s1=0..t1),s2=0..t2),t1,t2);

Construction of surfaces of correlation function Ky0 and Ky0n over a square [0,5] x [0,5]

> plot3d([Ky0(t1,t2),Ky0n(t1,t2)],t1=0..5,t2=0..5);

Calculation of the correlation functionat the corner point of a square

>evalf([Ky0(5,5), Ky0n(5,5)]);

[101.0401016, 114.7107820]

Determination of the correlation functionof a partial solution, corresponding to X(t) with correlation function Ky at ν = 1; 1/2

> Ky1:=unapply((Pi^2/4)*int(int(s1*s2*p(1,s1,t1)*p(1,s2,t2)* Kx(s1,s2),s1=0..t1),s2=0..t2),t1,t2);

$$Ky1 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} \frac{1}{4}(s1 \cdot s2 \cdot (BesselJ(1,t1) \cdot BesselY(1,s1) - BesselJ(1,s1) \cdot BesselY(1,t1)) \cdot \pi^2 \cdot t1 \cdot (BesselJ(1,t2) \cdot BesselY(1,s2) - BesselJ(1,s2) \cdot BesselY(1,t2)) \cdot t2 \cdot e^{-(s1-s2)^2}\right) ds1 ds2\right)\right)$$

>

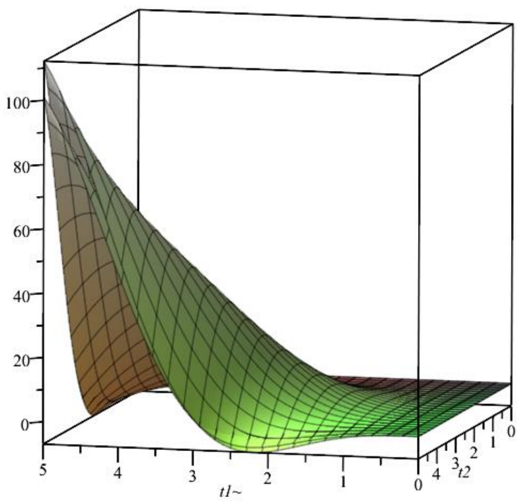

Figure 4

Ky12:=unapply((Pi^2/4)*int(int(s1*s2*p(1/2,s1,t1)*p(1/2, s2,t2)* Kx(s1,s2),s1=0..t1),s2=0..t2),t1,t2);

$$Ky12 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} \frac{1}{4}\left(s1 \cdot s2 \cdot \left(-\frac{2\sin(t1)\cdot\cos(s1)}{\pi\sqrt{t1}\sqrt{s1}} + \frac{2\sin(s1)\cdot\cos(t1)}{\pi\sqrt{s1}\sqrt{t1}}\right) \cdot \pi^2 \cdot t1 \cdot \left(-\frac{2\sin(t2)\cdot\cos(s2)}{\pi\sqrt{t2}\sqrt{s2}} + \frac{2\sin(s2)\cdot\cos(t2)}{\pi\sqrt{s2}\sqrt{t2}}\right) \cdot t2 \cdot e^{-(s1-s2)^2}\right) ds1 ds2\right)\right)$$

Construction of surfaces of correlation function Ky0, Ky1 over a square [0,5] x [0,5]

> plot3d([Ky0(t1,t2),Ky1(t1,t2)],t1=0..5,t2=0..5);

Calculation of thecorrelation function at the corner point of a square

>evalf([Ky0(5,5),Ky1(5,5),Ky12(5,5)]);

[101.0401016,112.1910409,103.6054206]

Calculation of the correlation function of the 1st and 2nd derivatives of X(t) (under the identifier Ku and Kv)

>Ku:=unapply(simplify(diff(Kx(s1,s2),s1,s2)),s1,s2);

$$Ku := (s1, s2) \mapsto -4(s1^2 - 2 \cdot s2 \cdot s1 + s2^2 - \frac{1}{2}) \cdot e^{-(s1-s2)^2}$$

>Kv:=unapply(simplify(diff(Ku(s1,s2),s1,s2)),s1,s2);

$$Kv := (s1, s2) \mapsto 16 \cdot (s1^4 - 4 \cdot s1^3 \cdot s2 + (6 \cdot s2^2 - 3) \cdot s1^2 + (-4 \cdot s2^3 + 6 \cdot s2) \cdot s1 + s2^4 - 3 \cdot s2^2 + \frac{3}{4})e^{-(s1-s2)^2}$$

Determination of the correlation function of a solution,corresponding to dX(t)/dt at ν = 0

> Kw0:=unapply((Pi^2/4)*int(int(s1*s2*p(0,s1,t1)*p(0,s2,t2)* Ku(s1,s2),s1=0..t1),s2=0..t2),t1,t2);

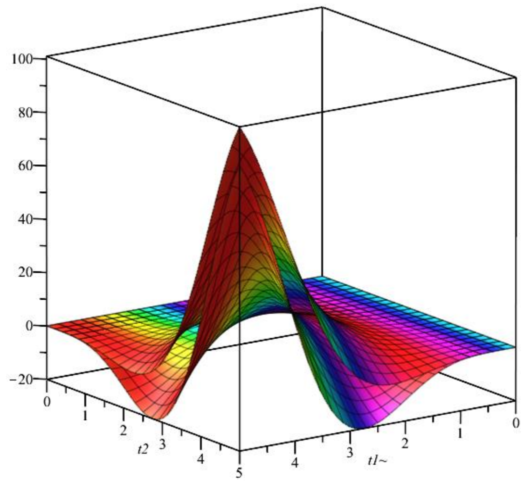

Figure 6

$$Kw0 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} -s1 \cdot s2 \cdot \right.\right.$$
$$\cdot \big(BesselJ(0,t1) \cdot BesselY(0,s1) - BesselJ(0,s1) \cdot$$
$$\cdot BesselY(0,t1)) \cdot \pi^2 \cdot t1 \cdot (BesselJ(0,t2) \cdot BesselY(0,s2) -$$
$$- BesselJ(0,s2) \cdot BesselY(0,t2)) \cdot$$
$$\left.\left.\cdot t2\left(s1^2 - 2 \cdot s1 \cdot s2 + s2^2 - \frac{1}{2}\right) \cdot e^{-(s1-s2)^2}\, ds1 ds2\right)\right)$$

Construction of surface Kw0

> plot3d(Kw0(t1,t2),t1=0..5,t2=0..5,color=sin(t1));

Construction of surfacesof Ky0 and Kw0 over a square

>
plot3d([Ky0(t1,t2),Kw0(t1,t2)],t1=0..5,t2=0..5,color=sin(t1));

Determination of the correlation function of a solution,corresponding to $\frac{d^2}{dt^2}X(t)$at ν = 0

>
Kz0:=unapply((Pi^2/4)*int(int(s1*s2*p(0,s1,t1)*p(0,s2,t2)*
Kv(s1,s2),s1=0..t1),s2=0..t2),t1,t2);

$$Kz0 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} 4 \cdot s1 \cdot s2 \cdot \right.\right.$$
$$\cdot \big(BesselJ(0,t1) \cdot BesselY(0,s1) - BesselJ(0,s1) \cdot$$
$$\cdot BesselY(0,t1)) \cdot \pi^2 \cdot t1 \cdot (BesselJ(0,t2) \cdot BesselY(0,s2) -$$
$$- BesselJ(0,s2) \cdot BesselY(0,t2)) \cdot$$
$$\cdot t2\big(s1^4 - 4 \cdot s1^3 \cdot s2 + (6 \cdot s2^2 - 3) \cdot s1^2 +$$
$$\left.\left. + (-4 \cdot s2^3 + 6 \cdot s2) \cdot s1 + s2^4 - 3 \cdot s2^2 + \frac{3}{4}\right) \cdot e^{-(s1-s2)^2}\, ds1 ds2\right)\right)$$

Table of variance values at ν = 0 at points t = 1, 2, 3, 4, 5 for k = 0, 1, 2 (under the identifier [Ky0 (i,i), Kw0 (i,i), Kz0 (i,i)])

> for i from 1 to 5 do evalf([Ky0(i,i),Kw0(i,i),Kz0(i,i)],3) od;

[0.124, 0.203, 0.992]
[4.38, 4.60, 14.0]
[22.0, 16.4, 34.8]
[52.0, 39.2, 76.5]
[101., 86.2, 149.]

Calculation of the correlation function of asolution,corresponding to $\frac{d}{dt}X(t)$ and $\frac{d^2}{dt^2}X(t)$at ν = 1

>
Kw1:=unapply((Pi^2/4)*int(int(s1*s2*p(1,s1,t1)*p(1,s2,t2)*
Ku(s1,s2),s1=0..t1),s2=0..t2),t1,t2);

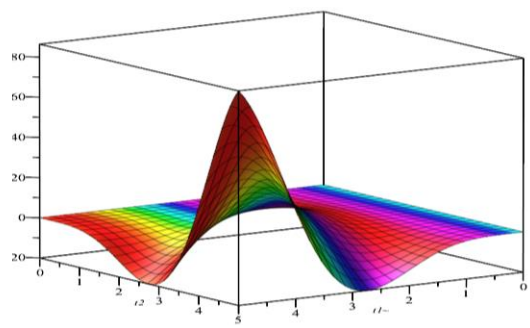

Figure 5

$$Kw1 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} -s1\cdot s2\cdot (BesselJ(1,t1\sim)\cdot BesselY(1,s1) - BesselJ(1,s1)\cdot BesselY(1,t1))\cdot \pi^2\cdot t1\cdot (BesselJ(1,t2)\cdot BesselY(1,s2) - BesselJ(1,s2)\cdot BesselY(1,t2))\cdot t2\left(s1^2 - 2\cdot s1\cdot s2 + s2^2 - \frac{1}{2}\right)\cdot e^{-(s1-s2)^2}\, ds1 ds2\right)\right)$$

```
>
Kz1:=unapply((Pi^2/4)*int(int(s1*s2*p(1,s1,t1)*p(1,s2,t2)*
Kv(s1,s2),s1=0..t1),s2=0..t2),t1,t2);
```

$$Kw1 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} 4\cdot s1\cdot s2\cdot (BesselJ(1,t1)\cdot BesselY(1,s1) - BesselJ(1,s1)\cdot BesselY(1,t1))\cdot \pi^2\cdot t1\cdot (BesselJ(1,t2)\cdot BesselY(1,s2) - BesselJ(1,s2)\cdot BesselY(1,t2))\cdot t2\cdot \left(s1^4 - 4\cdot s1^3\cdot s2 + (6\cdot s2^2 - 3)\cdot s1^2 + (-4\cdot s2^3 + 6\cdot s2)\cdot s1 + s2^4 - 3\cdot s2^2 + \frac{3}{4}\right)\cdot e^{-(s1-s2)^2}\, ds1 ds2\right)\right)$$

Construction of surfaces Ky1 and Kw1 over a square

```
>
plot3d([Ky1(t1,t2),Kw1(t1,t2)],t1=0..5,t2=0..5,color=sin(t1));
```

Table of variance values at ν =1 at points t = 1, 2, 3, 4, 5 for k = 0, 1, 2([Ky1 (i,i), Kw1 (i,i), Kz1 (i,i)])

```
> for i from 1 to 5 do evalf([Ky1(i,i),Kw1(i,i),Kz1(i,i)],3) od;
```

[0.215, 0.342, 1.64]

[7.12, 6.78, 18.5]

[31.5, 18.5, 31.5]

[62.8, 37.5, 63.2]

[112., 88.2, 143.]

Determination of the correlation function of a solution, corresponding to $\frac{d}{dt}X(t)$ and $\frac{d^2}{dt^2}X(t)$ at ν = 1/2

```
>
Kw12:=unapply((Pi^2/4)*int(int(s1*s2*p(1/2,s1,t1)*p(1/2,s2,t2)*
Ku(s1,s2),s1=0..t1),s2=0..t2),t1,t2);
```

$$Kw12 := (t1\ ,t2) \mapsto \frac{1}{4}\left(\pi^2 \cdot \left(\int_0^{t2}\int_0^{t1} -s1\cdot s2\cdot \left(\frac{2\sin(t1)\cdot\cos(s1)}{\pi\sqrt{t1}\sqrt{s1}} + \frac{2\sin(s1)\cdot\cos(t1)}{\pi\sqrt{s1}\sqrt{t1}}\right)\cdot \pi^2\cdot t1\cdot \left(-\frac{2\sin(t2)\cdot\cos(s2)}{\pi\sqrt{t2}\sqrt{s2}} + \frac{2\sin(s2)\cdot\cos(t2)}{\pi\sqrt{s2}\sqrt{t2}}\right)\cdot t2\cdot \left(s1^2 - 2\cdot s1\cdot s2 + s2^2 - \frac{1}{2}\right)\cdot e^{-(s1-s2)^2}\, ds1 ds2\right)\right)$$

```
>
Kz12:=unapply((Pi^2/4)*int(int(s1*s2*p(1/2,s1,t1)*p(1/2,s2,t2)*
Kv(s1,s2),s1=0..t1),s2=0..t2),t1,t2);
```

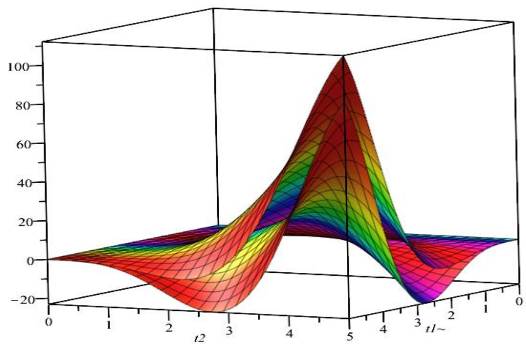

Figure 7

$$Kz12 := (t1\ , t2) \mapsto \frac{1}{4}\Bigg(\pi^2 \cdot \Bigg(\int_0^{t2}\int_0^{t1} 4 \cdot s1 \cdot s2 \cdot$$
$$\cdot\Bigg(-\frac{2\sin(t1)\cdot\cos(s1)}{\pi\sqrt{t1}\sqrt{s1}} +$$
$$+\frac{2\sin(s1)\cdot\cos(t1)}{\pi\sqrt{s1}\sqrt{t1}}\Bigg)\cdot\pi^2\cdot t1 \sim \cdot(-\frac{2\sin(t2)\cdot\cos(s2)}{\pi\sqrt{t2}\sqrt{s2}} +$$
$$\frac{2\sin(s2)\cdot\cos(t2)}{\pi\sqrt{s2}\sqrt{t2}})\cdot t2\big(s1^4 - 4\cdot s1^3\cdot s2 +$$
$$+(6\cdot s2^2 - 3)\cdot s1^2 + (-4\cdot s2^3 + 6\cdot s2)\cdot s1 +$$
$$+s2^4 - 3\cdot s2^2 + \frac{3}{4}\Big)\cdot e^{-(s1-s2)^2}\,ds1ds2\Bigg)\Bigg)$$

Table of variance values at ν = 1/2 at points t = 1, 2, 3, 4, 5 for k = 0,1, 2 ([Ky12 (i,i), Kw12 (i,i), Kz12 (i, i)])

```
> for i from 1 to 5 do
evalf([Ky12(i,i),Kw12(i,i),Kz12(i,i)],3)od ;
```

[0.140, 0.228, 1.11]
[4.85, 4.98, 14.7]
[23.7, 16.7, 33.5]
[54.2, 39.0, 73.8]
[104., 86.8, 147.]

Determination of the weight function p (ν, t, t1)

```
>restart;
>ps:=unapply((BesselJ(nu,t1)*BesselY(nu,t)-
BesselJ(nu,t)*BesselY(nu,t1)),nu,t,t1);
```

V CONCLUSION

It follows from the results obtained that for various types of correlation function of the input stationary random signal at the output of a dynamic system described by the Bessel equation, processes with correlation functions with similar characteristics are noted. Thus, some “smoothing“ of the input signal is implemented.

REFERENCES

[1] A.A. Sveshnikov, *Applied method of the theory of random functions*. Elsevier, 2014, 710 p.
[2] G.N. Watson, *A treatise on the theory of Bessel functions*. Forgotten Books, 2014, 818 p
[3] J. Vrbik, P. Vrbik, *Informal introduction to stochastic processes with Maple.* Springer, 2013, 287 p.
[4] MahmutParlar, *Interactive operations research with Maple*. Springer, 2012, 468 p.
[5] D. Bychkov, *Classification of two types of random process correlation function*. S.-Petersburg, 2012.
[6] A.M. Yaglom, *Correlation theory of stationary and related functions*. Springer, 2011, 258 p.

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